5.4 – Differential Equations

$y'' + 5y' + 6y = 0$

2nd-order DE

Definitions: A **differential equation** is an equation involving unknown functions and their derivatives.

The **order** of a differential equation is the order of the highest derivative it contains.

1st-order DE

A differential equation of the form $y' = ay$ has a **general solution** of the form $y = ce^{ax}$.

If $\frac{dy}{dx} = ay$, then $\frac{dy}{y} = a\,dx \Rightarrow \int \frac{dy}{y} = \int a\,dx$

$\Rightarrow \ln|y| = ax + C_1$

$\Rightarrow y = e^{ax+C_1} \Rightarrow y = e^{C_1}e^{ax}$

$y = ce^{ax}$

A condition which specifies the value of the general solution at a point is called an **initial condition**, and the problem of solving a differential equation subject to an initial condition is called an **initial-value problem**.

If we know, for instance, that $y(0) = P_0$, then $y = P_0 e^{ax}$.

A **constant coefficient first-order homogeneous linear system** has the form

$$\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n\\
y_2' &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n\\
&\vdots \qquad \vdots \quad \vdots \qquad \vdots\\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n
\end{aligned}$$

where $y_i = f_i(x)$ are functions to be determined, and the a_{ij}'s are constants.

This can be written in matrix notation as

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

or $\mathbf{y}' = A\mathbf{y}$.

The **trivial solution** to the above differential equation is $y_1 = y_2 = \ldots = y_n = 0$.

If we can find a matrix P that diagonalizes A, then we can use the diagonal matrix in solving the system. If P diagonalizes A, then we form $\vec{y} = P\vec{u}$, where $\vec{u}$ is an unknown vector of functions. Then

$$\vec{y}' = A\vec{y} \Rightarrow P\vec{u}' = A(P\vec{u})$$

$$\Rightarrow \vec{u}' = P^{-1}AP\vec{u}$$

This now has the form $\vec{u}' = D\vec{u}$, where D is diagonal

↖ $y' = A\vec{y}$

This has the form of a system that is solved using exponentials.

Recap: Imagine an unknown vector $\vec{u}$ of functions exists. Then find $\vec{u}$ and form $\vec{y} = P\vec{u}$.

Example:

a. Solve the system of differential equations.

$y_1' = 2y_1 + 3y_2$

$y_2' = 2y_1 + y_2$

b. Find the solution that satisfies the initial conditions $y_1(0) = 25$, $y_2(0) = 5$.

#8 Use the procedure in Exercise 7 to solve $y'' + y' - 12y = 0$.